OCR
Oxford Cambridge and RSA

Friday 23 June 2023 – Afternoon

A Level Further Mathematics A

Y545/01 Additional Pure Mathematics

Time allowed: 1 hour 30 minutes

9979637403

You must have:
- the Printed Answer Booklet
- the Formulae Booklet for A Level Further Mathematics A
- a scientific or graphical calculator

QP

INSTRUCTIONS
- Use black ink. You can use an HB pencil, but only for graphs and diagrams.
- Write your answer to each question in the space provided in the **Printed Answer Booklet**. If you need extra space use the lined pages at the end of the Printed Answer Booklet. The question numbers must be clearly shown.
- Fill in the boxes on the front of the Printed Answer Booklet.
- Answer **all** the questions.
- Where appropriate, your answer should be supported with working. Marks might be given for using a correct method, even if your answer is wrong.
- Give non-exact numerical answers correct to **3** significant figures unless a different degree of accuracy is specified in the question.
- The acceleration due to gravity is denoted by $g\,\text{m}\,\text{s}^{-2}$. When a numerical value is needed use $g = 9.8$ unless a different value is specified in the question.
- Do **not** send this Question Paper for marking. Keep in the centre or recycle it.

INFORMATION
- The total mark for this paper is **75**.
- The marks for each question are shown in brackets **[ ]**.
- This document has **4** pages.

ADVICE
- Read each question carefully before you start your answer.

1 The surface S is defined for all real x and y by the equation $z = x^2 + 2xy$. The intersection of S with the plane Π gives a section of the surface. On the axes provided in the Printed Answer Booklet, sketch this section when the equation of Π is each of the following.

(a) $x = 1$ **[2]**

(b) $y = 1$ **[2]**

2 A curve has equation $y = \sqrt{1+x^2}$, for $0 \leqslant x \leqslant 1$, where both the x- and y-units are in cm. The area of the surface generated when this curve is rotated fully about the x-axis is $A\,\text{cm}^2$.

(a) Show that $A = 2\pi \int_0^1 \sqrt{1+kx^2}\,\mathrm{d}x$ for some integer k to be determined. **[4]**

A small component for a car is produced in the shape of this surface. The curved surface area of the component must be $8\,\text{cm}^2$, accurate to within one percent. The engineering process produces such components with a curved surface area accurate to within one half of one percent.

(b) Determine whether all components produced will be suitable for use in the car. **[2]**

3 The points A and B have position vectors $\mathbf{a} = \mathbf{i} + p\mathbf{j} + q\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ respectively, relative to the origin O.

(a) Determine the value of p and the value of q for which $\mathbf{a} \times \mathbf{b} = 2\mathbf{i} + 6\mathbf{j} - 11\mathbf{k}$. **[3]**

(b) The point C has coordinates (d, e, f) and the tetrahedron $OABC$ has volume 7.

(i) Using the values of p and q found in part **(a)**, find the possible relationships between d, e and f. **[2]**

(ii) Explain the geometrical significance of these relationships. **[2]**

4 The sequence $\{A_n\}$ is given for all integers $n \geqslant 0$ by $A_n = \dfrac{I_{n+2}}{I_n}$, where $I_n = \int_0^{\frac{1}{2}\pi} \cos^n x\,\mathrm{d}x$.

- Show that $\{A_n\}$ increases monotonically.
- Show that $\{A_n\}$ converges to a limit, A, whose exact value should be stated. **[7]**

5 **(a)** The group G consists of the set $S = \{1, 9, 17, 25\}$ under $\times_{32}$, the operation of multiplication modulo 32.

(i) Complete the Cayley table for G given in the Printed Answer Booklet. **[2]**

(ii) Up to isomorphisms, there are only two groups of order 4.

- C_4, the cyclic group of order 4
- K_4, the non-cyclic (Klein) group of order 4

State, with justification, to which of these two groups G is isomorphic. **[2]**

(b) **(i)** List the odd quadratic residues modulo 32. **[2]**

(ii) Given that n is an odd integer, prove that $n^6 + 3n^4 + 7n^2 \equiv 11 \pmod{32}$. **[4]**

6 The surface S has equation $z = x\sin y + \frac{y}{x}$ for $x > 0$ and $0 < y < \pi$.

(a) Determine, as a function of x and y, the determinant of $\mathbf{H}$, the Hessian matrix of S. **[6]**

(b) Given that S has just one stationary point, P, use the answer to part **(a)** to deduce the nature of P. **[2]**

(c) The coordinates of P are (α, β, γ).

Show that β satisfies the equation $\beta + \tan\beta = 0$. **[3]**

7 Binet's formula for the nth Fibonacci number is given by $F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n)$ for $n \geqslant 0$, where α and β (with $\alpha > 0 > \beta$) are the roots of $x^2 - x - 1 = 0$.

(a) Write down the values of $\alpha + \beta$ and $\alpha\beta$. **[1]**

(b) Consider the sequence $\{S_n\}$, where $S_n = \alpha^n + \beta^n$ for $n \geqslant 0$.

(i) Determine the values of S_2 and S_3. **[3]**

(ii) Show that $S_{n+2} = S_{n+1} + S_n$ for $n \geqslant 0$. **[2]**

(iii) Deduce that S_n is an integer for all $n \geqslant 0$. **[1]**

(c) A student models the terms of the sequence $\{S_n\}$ using the formula $T_n = \alpha^n$.

(i) Explain why this formula is unsuitable for every $n \geqslant 1$. **[1]**

(ii) Considering the cases n even and n odd separately, state a modification of the formula $T_n = \alpha^n$, other than $T_n = \alpha^n + \beta^n$, such that $T_n = S_n$ for all $n \geqslant 1$. **[2]**

8 Let $f(n)$ denote the base-n number 2121_n where $n \geqslant 3$.

(a) **(i)** For each $n \geqslant 3$, show that $f(n)$ can be written as the product of two positive integers greater than 1, $a(n)$ and $b(n)$, each of which is a function of n. **[2]**

(ii) Deduce that $f(n)$ is always composite. **[1]**

(b) Let h be the highest common factor of $a(n)$ and $b(n)$.

(i) Prove that h is either 1 or 5. **[4]**

(ii) Find a value of n for which $h = 5$. **[2]**

9 The set C consists of the set of all complex numbers excluding 1 and -1. The operation $\oplus$ is defined on the elements of C by $a \oplus b = \dfrac{a+b}{ab+1}$ where $a, b \in C$.

(a) Determine the identity element of C under $\oplus$. **[2]**

(b) For each element x in C show that it has an inverse element in C. **[2]**

(c) Show that $\oplus$ is associative on C. **[3]**

(d) Explain why $(C, \oplus)$ is not a group. **[1]**

(e) Find a subset, D, of C such that $(D, \oplus)$ is a group of order 3. **[3]**

END OF QUESTION PAPER